UWThPh-1999-37
ESI-720-1999
math-ph/9906020
June 1, 1999

Anyons and the Bose-Fermi duality in the finite-temperature Thirring model[⋆]

N. Ilieva[∗,♯] and W. Thirring

Institut für Theoretische Physik
Universität Wien
and
Erwin Schrödinger International Institute
for Mathematical Physics

Abstract

Solutions to the Thirring model are constructed in the framework of algebraic quantum field theory. It is shown that for all positive temperatures there are fermionic solutions only if the coupling constant is $\lambda = \sqrt{2(2n+1)\pi}$, $n \in \mathbf{N}$. These fermions are inequivalent and only for $n = 1$ they are canonical fields. In the general case solutions are anyons. Different anyons (which are uncountably many) live in orthogonal spaces and obey dynamical equations (of the type of Heisenberg's "Urgleichung") characterized by the corresponding values of the statistic parameter. Thus statistic parameter turns out to be related to the coupling constant λ and the whole Hilbert space becomes non-separable with a different "Urgleichung" satisfied in each of its sectors. This feature certainly cannot be seen by any power expansion in λ. Moreover, since the latter is tied to the statistic parameter, it is clear that such an expansion is doomed to failure and will never reveal the true structure of the theory.

The correlation functions in the temperature state for the canonical dressed fermions are shown by us to coincide with the ones for the bare fields, that is in agreement with the uniqueness of the τ-KMS state over the CAR algebra (τ being the shift automorphism). Also the α-anyon two-point function is evaluated and for scalar field it reproduces the result that is known from the literature.

PACS codes: 03.70.+k, 11.10.Wx, 11.10.Kk, 05.30.-d

[⋆] Work supported in part by "Fonds zur Förderung der wissenschaftlichen Forschung in Österreich" under grant P11287–PHY, *to be published in Theor. Math. Phys.*

[∗] On leave from Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul.Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria

[♯] E–mail address: ilieva@ap.univie.ac.at

It is an honour for us to present this paper to the 30th anniversary of "Theoretical and Mathematical Physics". With its scientific reputation and high standards the journal takes a valuable place in the heritage of its founder, N.N. Bogoliubov — one of the outstanding scientists of this century, and we wish this to be kept also in the future.

1 Introduction

After T.D. Lee had constructed a model of a soluble QFT [1] many people tried to find other examples; but to solve a nontrivial relativistic QFT seemed out of the question. The idea that Bethe's ansatz [2] could be successfully used to solve also Heisenberg's "Urgleichung" [3] reduced to one space one time dimension then led to a soluble relativistic field theory — the Thirring model [4]. During the years, this model has not only been extensively studied but has also been actively used for analysis, testing and illustration of various phenomena in two-dimensional field theories.

It is not our purpose to review the enormous literature on the subject but we rather focus on the very starting point — Heisenberg's Urgleichung. With no bosons present in it at all, it represents the ultimate version of the opinion that fermions should enter the basic formalism of the fundamental theory of elementary particles that is usually taken for granted.

The opposite point of view, namely that a theory including only observable fields, necessarily uncharged bosons, is capable of describing evolution and symmetries of a physical system, being the kernel of algebraic approach to QFT [5], also enjoys an enthusiastic support. Actually, the question which is thus posed and which is of principal importance is whether and in which cases definite conclusions about the time evolution and symmetries of charged fields can be drawn from the knowledge about the observables that is gained through experiment.

As we will see, there is no possibility to judge this matter on the basis of the model in question, since both formulations can be equally well used to construct the physically relevant objects — the dressed fermions.

In any case, before claiming that an "Urgleichung" of the type

$$\partial\!\!\!/\psi(x) = \lambda\psi(x)\bar{\psi}(x)\psi(x) \tag{1.1}$$

determines the whole Universe one should see whether it determines anything mathematically and it is our aim in the present paper to discuss the elements needed to make its solution well defined. In fact we shall first consider only one chiral component and we shall restrict ourselves to the two-dimensional spacetime, so that this component depends only on one light cone coordinate. Also the bose-fermi duality takes place there and we want to make use of it. This phenomenon amounts to the fact that in certain models formal functions of fermi fields can be written that have vacuum expectation values and statistics of bosons and vice versa. The equivalence is understood within perturbation theory: the perturbation series for the so-related theories are term-by-term equivalent

(they may perfectly well exist even if the models are not exactly solvable or if their physical sensibility is doubtful).

There are two facts which make such a duality possible. First comes the main reason why soluble fermion models exist in two dimensions, that is that fermion currents can be constructed as "fields" acting on the representation space for the fermions. Also, the "bosons into fermions" programme rests on the fact that bosons in question are just the currents and fermions are essentially determined by their commutation relations with them. Second comes the observation which has been made in the pioneering works by Jordan [6] and Born [7]: due to the unboundedness from below of the free-fermion Hamiltonian the fermion creation and annihilation operators must undergo what we should call now a Bogoliubov transformation. Thus the stability of the system is achieved but in addition an anomalous term (later called "Schwinger term") appeares in the current commutator, that in turn enables the "bosonization".

The bose-fermi duality is actually well established when the construction of bosons out of fermions is considered so that consistent expressions exist for the fermion bilinears that are directly related to the observables of the theory.

The problem of rigorous definitions of operator valued distributions and eventually operators having the basic properties of fermions by taking functions of bosonic fields is rather more delicate. On the level of operator valued distributions solutions have been given by Dell'Antonio et al.[8] and Mandelstam [9] and on the level of operators in a Hilbert space — by Carey and collaborators [10, 11] and in a Krein space by Acerbi, Morchio and Strocchi [12].

Thus our goal is to give in one and the same setting a precise meaning to the following three ingredients

$$\begin{array}{llll} (a) & [\,\psi^*(x),\psi(x')\,]_+ = \delta(x-x'), \qquad [\,\psi(x),\psi(x')\,]_+ = 0 & \text{CAR} & \\ (b) & j(x) = \psi^*(x)\psi(x) & \text{Current} & (1.2) \\ (c) & \frac{1}{i}\frac{d}{dx}\psi(x) = \lambda j(x)\psi(x) & \text{Urgleichung} & \end{array}$$

We shall approach it by constructing a series of algebraic inclusions, starting from the CAR-algebra of bare fermions. Eq.(1.2c) involves (derivatives of) objects which are according to (1.2a) rather discontinuous. Therefore it is expedient to pass right away to the level of operators in Hilbert space since the variety of topologies there provides a better control over the limiting procedures. In general norm convergence can hardly be hoped for but we have to strive at least for strong convergence such that the limit of the product is the product of the limits. With $\psi_f = \int_{-\infty}^{\infty} dx f(x)\psi(x)$, (1.2a) becomes

$$[\,\psi_f^*, \psi_g\,]_+ = \langle f|g\rangle \tag{1.3}$$

for $f \in L^2(\mathbf{R})$ and $\langle .|.\rangle$ the scalar product in $L^2(\mathbf{R})$. This shows that ψ_f's are bounded and form the C^*-algebra CAR. There the translations $x \to x+t$ give an automorphism τ_t and we shall use the corresponding KMS-states ω_β and the associated representation π_β to extend CAR. Though there $j=\infty$, one can give a meaning to j as a strong limit

in $\mathcal{H}_\beta$ by smearing $\psi(x)$ over a region ε to $\psi_\varepsilon(x)$ and then defining

$$j_f = \int dx f(x) \lim_{\varepsilon\to 0} \left(\psi^*_\varepsilon(x)\psi_\varepsilon(x) - \omega_\beta(\psi^*_\varepsilon(x)\psi_\varepsilon(x))\right), \qquad f : \mathbf{R} \to \mathbf{R}$$

These limits exist in the strong resolvent sense and define self-adjoint operators with a multiplication law

$$e^{ij_f} e^{ij_g} = e^{\frac{i}{8\pi}\int dx (f(x)g'(x) - f'(x)g(x))} e^{ij_{f+g}}\,. \tag{1.4}$$

Thus the current algebra $\mathcal{A}_c$ is determined. Its Weyl structure is the same for all positive β and ω_β extends to $\mathcal{A}_c$.

To construct the interacting fermions which on the level of distributions look like

$$\Psi(x) = Z\, e^{i\lambda \int_{-\infty}^{x} dx' j(x')} \overset{?}{=} \lim_{\varepsilon\to 0_+} \lim_{R\to\infty} \Psi_{\varepsilon,R}(x)$$

(with some renormalization constant Z) poses both infrared ($R \to \infty$) and ultraviolet ($\varepsilon \to 0$) problems. For

$$\Psi_{\varepsilon,R}(x) = e^{i\lambda \int dx' (\varphi_\varepsilon(x-x') - \varphi_\varepsilon(x-x'+R)) j(x')}, \qquad \varphi_\varepsilon(x) := \begin{cases} 1 & \text{for } x \le -\varepsilon \\ -x/\varepsilon & \text{for } -\varepsilon \le x \le 0 \\ 0 & \text{for } x \ge 0 \end{cases}$$

neither the limit $R \to \infty$ nor the limit $\varepsilon \to 0$ exist even as weak limits in $\mathcal{H}_\beta$. So an extension of $\pi_\beta(\mathcal{A}_c)''$ is needed to accomodate such a kind of objects.

There are two equivalent ways of handling the infrared problem. Since the automorphism generated by the unitaries $\Psi_{\varepsilon,R}(x)$ for $R \to \infty$ converges to a limit γ, one can form with it the crossed product $\bar{\mathcal{A}}_c = \mathcal{A}_c \overset{\gamma}{\bowtie} \mathrm{Z}$, so that in $\bar{\mathcal{A}}_c$ there are unitaries with the properties which the limit should have [13, 14]. On the other hand, the symplectic form in (1.4) and the state ω_β can be defined for the limiting element $\Psi_\varepsilon(x)$ and this we shall do in what follows. The former route will be then discussed in Appendix A.

In any case $\bar{\mathcal{H}}_\beta$ assumes a sectorial structure, the subspaces $\mathcal{A}_c \prod_{i=1}^{n} \Psi_\varepsilon(x_i)|\Omega\rangle$ for different n are orthogonal and thus may be called n-fold charged sectors. The $\Psi_\varepsilon(x)$'s have the property that for $|x_i - x_j| > 2\varepsilon$ they obey anyon statistics with parameter λ^2 and an Urgleichung (1.2c) where $j(x)$ is averaged over a region of lenght ε below x.

Then, by removing the ultraviolet cut-off the sectors abound and the subspaces $\mathcal{A}_c\Psi(x)|\Omega\rangle$ become orthogonal for different x, so $\bar{\mathcal{H}}_\beta$ becomes non-separable. To get canonical fields of the type (1.3) one has to combine $\varepsilon \to 0_+$ with a field renormalization $\Psi_\varepsilon \to \varepsilon^{-1/2}\Psi_\varepsilon$ such that

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-1/2} \int dx f(x) \Psi_\varepsilon(x) = \Psi_f$$

converge strongly in $\bar{\mathcal{H}}_\beta$ and satisfy (1.2c) in sense of distributions.

The current (1.2b) has been constructed with the bare fermions ψ and is obviously sensitive to the infinite renormalization in the dressed field Ψ. Therefore it is better to replace (1.2b) by the requirement that j_f generates a local gauge transformation. Indeed,

$$e^{ij_f}\Psi_g e^{-ij_f} = \Psi_{e^{if}g} \tag{1.5}$$

holds and in this sense (1.2b) is also satisfied.

However, the objects so constructed are in general anyons and only for particular values of the coupling constant, $\lambda = \sqrt{2(2n+1)\pi},\ n \in \mathbf{N}$, they are fermions, so that the coupling constant is tied to the statistic parameter. Thus we find that there is indeed some magic about the Urgleichung inasmuch as on the quantum level it allows fermionic solutions by this construction only for isolated values of the coupling constant λ whereas classically $\Psi(x) = Z\, e^{i\lambda \int_{-\infty}^{x} dx' j(x')}$ solves (1.2c) for any λ. This feature can certainly not be seen by any power expansion in λ.

The scheme presented here means, that the dressed fermions obtained for special values of λ (and distinct from the bare ones) can be constructed either from bare fermions or directly from the current algebra, so no priority might be asigned to either of the two formulations, bosonic or fermionic. To make this statement precise, the correlation functions arrising in both cases have to be compared. As we shall see, for canonical fermions they do coincide that is in agreement with the uniqueness of the τ-KMS state over the CAR algebra. We shall also discuss the thermal correlators for the anyonic fields, in particular, we shall find an agreement with the recent result in [15] for the scalar-field case.

By a *symmetry* of a physical system an automorphism α of the algebra $\mathcal{A}$ which describes it is understood. The algebraic chain of inclusions we construct gives an example of a *symmetry destruction*, that is, for a given extension $\mathcal{B}$ of the algebra $\mathcal{A}$, $\mathcal{B} \supset \mathcal{A}$, $\nexists \beta \in \mathrm{Aut}\,\mathcal{B} : \beta|_{\mathcal{A}} = \alpha$ for some $\alpha \in \mathrm{Aut}\,\mathcal{A}$. This phenomenon is related to the spontaneous collapse of a symmetry [16] and in contrast to the spontaneous symmetry breaking [17], it cannot occur in a finite-dimensional Hilbert space.

2 Bosons out of fermions: the CAR-algebra, its KMS-states and associated v. Neumann algebras

Let us consider the C*-algebra $\mathcal{A}^l$ formed by the bounded operators

$$\psi_f = \int_{-\infty}^{\infty} dx \psi(x) f(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \tilde{\psi}(p) \tilde{f}(p), \qquad \tilde{f}(p) = \int_{-\infty}^{\infty} dx\, e^{ipx} f(x) \tag{2.1}$$

with $\psi(x)$, $x \in \mathbf{R}$, being operator-valued distributions which satisfy

$$[\psi^*(x), \psi(x')]_+ = \delta(x - x'), \tag{2.2}$$

so, describing the left movers (we have asigned a superscript to the relevant quantities, x stands for $x-t$) and $f \in L^2(\mathbf{R})$. This algebra is characterized by

$$[\psi_f^*, \psi_g]_+ = \langle f|g\rangle = \int dx f^*(x)g(x). \tag{2.3}$$

Translations τ_t define an automorphism of $\mathcal{A}^l$

$$\tau_t\psi_f = \psi_{f_t}, \quad f_t(x) = f(x-t). \tag{2.4}$$

$\mathcal{A}^l$ inherits the norm from $L^2(\mathbf{R})$ such that τ_t is (pointwise) normcontinuous in t and even normdifferentiable for the dense set of f's for which

$$\lim_{\delta\to 0_+} \frac{f(x+\delta)-f(x)}{\delta} = f'(x)$$

exists in $L^2(\mathbf{R})$

$$\frac{d}{dt}\,\tau_t\psi_f\Big|_{t=0} = -\psi_{f'}. \tag{2.5}$$

The τ-KMS-states over $\mathcal{A}^l$ are given by

$$\omega_\beta(\psi_f^*\psi_g) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\widetilde{f}^*(p)\widetilde{g}(p)}{1+e^{\beta p}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2\pi} \int \frac{dxdx' f^*(x)g(x')}{i(x-x')-n\beta+\varepsilon}, \qquad \varepsilon\to 0_+, \tag{2.6}$$

$$\omega_\beta(\psi_g\psi_f^*) = \omega_\beta(\psi_f^*\tau_{i\beta}\psi_g).$$

With each ω_β are associated a representation π_β with cyclic vector $|\Omega\rangle$, $\omega_\beta(a) = \langle\Omega|a|\Omega\rangle$ in $\mathcal{H}_\beta = \overline{\mathcal{A}^l|\Omega\rangle}$ and a v. Neumann algebra $\pi_\beta(\mathcal{A}^l)''$. It contains the current algebra $\mathcal{A}^l_c$ which gives the formal expression $j(x) = \psi^*(x)\psi(x)$ a precise meaning.

To show this, let us recall two lemmas (for the proofs see [14]) which make the whole construction transparent:

Lemma (2.7)
If the kernel $K(k,k') : \mathbf{R}^2 \to C$ is as operator ≥ 0 and trace class ($K(k,k) \in L^1(\mathbf{R})$), then $\forall\, \beta \in \mathbf{R}^+$

$$\begin{aligned}
\lim_{M\to\pm\infty} B_M &:= \lim_{M\to\pm\infty} \frac{1}{(2\pi)^2} \int dkdk' K(k,k')\widetilde{\psi}^*(k+M)\widetilde{\psi}(k'+M) = \\
&= \frac{1}{(2\pi)^2} \int dkdk' \lim_{M\to\pm\infty} K(k,k')\omega_\beta(\widetilde{\psi}^*(k+M)\widetilde{\psi}(k'+M)) = \\
&= \begin{cases} \frac{1}{2\pi}\int dk\ K(k,k) & \text{for } M\to+\infty \\ 0 & \text{for } M\to-\infty \end{cases}
\end{aligned}$$

in the strong sense in $\mathcal{H}_\beta$.

However, if $\int|K|^2$ keeps increasing with M, then $B_M - \langle B_M\rangle$ may nevertheless tend to an (unbounded) operator.

Lemma (2.8)
If

$$B_M = \frac{1}{(2\pi)^2} \int dk dk' \tilde{f}(k-k') \Theta(M - |k|) \Theta(M - |k'|) \tilde{\psi}^*(k) \psi(k')$$

with $\tilde{f}$ decreasing faster than an exponential and being the Fourier transform of a positive function, then the difference $B_M - \omega_\beta(B_M)$ is a strong Cauchy sequence for $M \to \infty$ on a dense domain on $\mathcal{H}_\beta$.

Remarks (2.9)

1. (2.7) substantiates the feeling that for $k > 0$ most levels are empty and for $k < 0$ most are full.
2. B_M is a positive operator and by diagonalizing K one sees

$$\|B_M\| = \|K\|_1 = \frac{1}{2\pi} \int dk\ K(k,k).$$

3. As just mentioned, $\|B_M\| < 2M\tilde{f}(0)$ and $f(x) \geq 0$ is not a serious restriction since any function is a linear combination of positive functions.
4. Since the limit j_f is unbounded the convergence is not on all of $\mathcal{H}_\beta$, however since for the limit j_f holds $\tau_{i\beta} j_f = j_{e^{\beta p} f}$, the dense domain is invariant under j_f. Thus we have strong resolvent convergence which means that bounded functions of B_M converge strongly. Also the commutator of the limits is the limit of the commutator.

Thus we conclude that the limit exists and is selfadjoint on a suitable domain. We shall write it formally

$$\begin{aligned} j_f &= \lim_{M \to \pm\infty} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk dk'\, K(k,k') \tilde{\psi}^*(k+M) \tilde{\psi}(k'+M) \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk dk\ \tilde{f}(k-k') : \tilde{\psi}(k)^* \tilde{\psi}(k') : \end{aligned} \tag{2.10}$$

Next we show that the currents so defined satisfy the CCR with a suitable symplectic form σ [6, 18].

Theorem (2.11)

$$[j_f, j_g] = i\sigma(f,g) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2}\, p\tilde{f}(p)\tilde{g}(-p) = \frac{i}{4\pi} \int_{-\infty}^{\infty} dx (f'(x)g(x) - f(x)g'(x)).$$

Proof: For the distributions $\widetilde{\psi}(k)$ we get algebraically

$$[\widetilde{\psi}^*(k)\widetilde{\psi}(k'), \widetilde{\psi}^*(q)\widetilde{\psi}(q')] = 2\pi \left[\widetilde{\psi}^*(k)\widetilde{\psi}(q')\delta(q-k') - \widetilde{\psi}^*(q)\widetilde{\psi}(k')\delta(k-q')\right]$$

and for the operators after some change of variables

$$\frac{1}{(2\pi)^3}\int dkdpdp' \widetilde{f}(p)\widetilde{g}(p')\widetilde{\psi}^*(p+p'+k)\widetilde{\psi}(k)\Theta(M-|k|)\Theta(M-|p+p'+k|)\cdot$$
$$\cdot\left[\Theta(M-|p'+k|) - \Theta(M-|p+k|)\right].$$

For fixed p and p' and $M \to \infty$ we see that the allowed region for k is contained in $(M-|p|-|p'|, M)$ and $(-M, -M+|p|+|p'|)$. Upon $k \to k \pm M$ we are in the situation of (2.7), thus we see that the commutator of the currents (2.10) is bounded uniformly in M if $\widetilde{f}$ and $\widetilde{g}$ decay faster than exponentials and converges to the expectation value. This gives finally

$$\int_{-\infty}^{\infty}\frac{dp}{(2\pi)^2}\ \widetilde{f}(p)\widetilde{g}(-p)\int dk\ \Theta(M-|k|)\left[\Theta(M-|k-p|) - \Theta(M-|k+p|)\right]\frac{1}{1+e^{\beta k}}$$
$$\stackrel{M\to\infty}{\longrightarrow}\int_{-\infty}^{\infty}\frac{dp}{(2\pi)^2}\ p\widetilde{f}(p)\widetilde{g}(-p).$$

Remarks (2.12)

1. Since the j_f's satisfy the CCR they cannot be bounded and it is better to write (2.11) in the Weyl form for the associated unitaries
$$e^{ij_f}\ e^{ij_g} = e^{\frac{i}{2}\sigma(g,f)}\ e^{ij_{f+g}} = e^{i\sigma(g,f)}\ e^{ij_g}\ e^{ij_f}.$$

2. The currents j_f are selfadjoint, so the unitaries $e^{i\alpha j_f}$ generate one-parameter groups — the local gauge transformations
$$e^{-i\alpha j_f}\ \psi_g\ e^{i\alpha j_f} = \psi_{e^{i\alpha f}g}.$$

3. The state ω_β can be extended to $\bar{\omega}_\beta$ over $\pi_\beta(\mathcal{A}^l)''$ and τ_t to $\bar{\tau}_t$, $\bar{\tau}_t \in \mathrm{Aut}\ \pi_\beta(\mathcal{A}^l)''$ with $\bar{\tau}_t\, j_f = j_{f_t}$. Furthermore $\bar{\omega}_\beta$ is $\bar{\tau}$-KMS and is calculated to be ([14], see also [15])
$$\bar{\omega}_\beta(e^{ij_f}) = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dp}{(2\pi)^2}\frac{p}{1-e^{-\beta p}}|\widetilde{f}(p)|^2\right].$$

4. A physically important symmetry of the algebra $\mathcal{A}^l$, the parity P,
$$P \in \mathrm{Aut}\ \mathcal{A}^l, \quad P\psi_f = \psi_{Pf}, \quad Pf(x) = f(-x)$$
is destroyed in π_β, since
$$[j(x), j(x')] = -\frac{i}{2\pi}\,\delta'(x-x')$$
is not invariant under $j(x) \to j(-x)$. Thus $P \notin \mathrm{Aut}\ \pi_\beta(\mathcal{A}^l)''$ and $\bar{\omega}_\beta$ is not P-invariant.

5. The extended shift automorphism $\bar{\tau}_t$ is not only strongly continuous but for suitable f's also differentiable in t (strongly on a dense set in $\mathcal{H}_\beta$)

$$\frac{1}{i}\frac{d}{dt}\bar{\tau}_t e^{ij_f} = \left[j_{f'_t} + \frac{1}{2}\sigma(f_t, f'_t)\right] e^{ij_{f_t}} = e^{ij_{f_t}}\left[j_{f'_t} - \frac{1}{2}\sigma(f_t, f'_t)\right] = \frac{1}{2}\left[j_{f'_t}\, e^{ij_{f_t}} + e^{ij_{f_t}} j_{f'_t}\right].$$

6. The symplectic structure is formally independent on β [19], however for $\beta < 0$ it changes its sign, $\sigma \to -\sigma$, and for $\beta = 0$ (the tracial state) it becomes zero.

Thus starting from a CAR-algebra $\mathcal{A}^l$, we identified in $\pi_\beta(\mathcal{A}^l)''$ bosonic fields — the currents, which satisfy CCR's. The crucial ingredient needed was the appropriately chosen state. Here we have used the KMS-state (which is unique for the CAR algebra). Another possibility would be to introduce the Dirac vacuum (filling all negative energy levels in the Dirac sea). This is what has been done in the thirties [6, 7], in order to achieve stability for a fermion system, and recovered later by Mattis and Lieb [20] in the context of the Luttinger model. Thus as an additional effect the appearance of an anomalous term in the current commutator (later called Schwinger term) had been discovered that actually enables bosonization of these two-dimensional models.

3 Extensions of $\mathcal{A}_c$: fermions out of bosons

So far $\mathcal{A}_c^l$ was defined for j_f's with $f \in C_0^\infty$, for instance. The algebraic structure is determined by the symplectic form $\sigma(f,g)$ (2.11) which is actually well defined also for the Sobolev space, $\sigma(f,g) \to \sigma(\bar{f},\bar{g})$, $\bar{f},\bar{g} \in H_1$, $H_1 = \{f : f, f' \in L^2\}$. Also $\bar{\omega}_\beta$ can be extended to H_1, since $\bar{\omega}_\beta(e^{ij_{\bar{f}}}) > 0$ for $\bar{f} \in H_1$. The anticommuting operators we are looking for are of the form e^{ij_f}, with $f(x) = 2\pi\Theta(x_0 - x) \notin H_1$. Still one can give $\sigma(f,g)$ a meaning for such an f. However, the corresponding state ω_β exhibits singular behaviour for both $p \to 0$ and $p \to \infty$, so that

$$\omega_\beta(e^{ij_\Theta}) = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} \frac{dp}{p(1-e^{-\beta p})}\right] = 0$$

and thus such an operator would act in $\mathcal{H}_\beta$ as zero. Therefore an approximation of Θ by functions from H_1 would result in unitaries that converge weakly to zero.

This situation can be visualized by the following

Example (3.1)
Consider the H_1-function $\Phi_{\delta,\varepsilon}(x)$,

$$\Phi_{\delta,\varepsilon}(x) := \varphi_\varepsilon(x) - \varphi_\varepsilon(x+\delta) \in H_1,$$

with

$$\varphi_\varepsilon(x) := \begin{cases} 1 & \text{for } x \le -\varepsilon \\ -x/\varepsilon & \text{for } -\varepsilon \le x \le 0 \\ 0 & \text{for } x \ge 0 \end{cases}$$

as an approximation to the step function,

$$\lim_{\substack{\delta\to\infty\\ \varepsilon\to 0}} \Phi_{\delta,\varepsilon}(x) = \Theta(x).$$

Then

$$\widetilde{\Phi}_{\delta,\varepsilon}(p) = \frac{1-e^{ip\varepsilon}}{\varepsilon p^2}(1-e^{ip\delta})$$

and

$$\|\Phi_{\delta,\varepsilon}\|_\beta^2 = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{p}{1-e^{-\beta p}}|\widetilde{\Phi}(p)|^2 = 16\int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{p}{1-e^{-\beta p}}\frac{\sin^2 p\varepsilon/2}{\varepsilon^2 p^4}\sin^2 p\delta/2,$$

$$\|\Phi\|_\beta^2 \geq c\int_0^{1/\delta} dp\ \delta^2 = c\delta$$

for β/δ, $\varepsilon/\delta \ll 1$ and c a constant. Thus for $\delta\to\infty$, $\|\Phi_\delta\|_\beta \to \infty$. Also $\|\Phi_\delta - f\|_\beta \to \infty$ since

$$\|\Phi_\delta - f\|_\beta \geq \|\Phi_\delta\|_\beta - \|f\|_\beta \to \infty \qquad \forall\ \|f\|_\beta < \infty$$

and thus

$$|\langle\Omega|e^{-ij_f}\ e^{ij_{\Phi_\delta}}|\Omega\rangle| = e^{-\frac{1}{2}\|\Phi_\delta - f\|_\beta^2} \to 0.$$

But $e^{ij_f}|\Omega\rangle$, $\|f\|_\beta < \infty$, is total in $\mathcal{H}_\beta$ and thus $e^{ij_{\Phi_\delta}}|\Omega\rangle$ and therefore $e^{ij_{\Phi_\delta}}$ goes weakly to zero.

However the automorphism

$$e^{ij_f} \to e^{-ij_{\Phi_\delta}}e^{ij_f}e^{ij_{\Phi_\delta}} = e^{i\sigma(\Phi_\delta,f)}e^{ij_f}$$

converges since

$$\sigma(f,\Phi_\delta) = -\frac{1}{2\pi\varepsilon}\left(\int_{-\varepsilon}^{0} - \int_{-\varepsilon-\delta}^{-\delta}\right) dx\ f(x) \overset{\delta\to\infty}{\longrightarrow} -\frac{1}{2\pi\varepsilon}\int_{-\varepsilon}^{0} dx\ f(x) \overset{\varepsilon\to 0}{\longrightarrow} -\frac{1}{2\pi}f(0).$$

The divergence of $\|\Phi_{\delta,\varepsilon}\|$ is related to the well-known infrared problem of the massless scalar field in (1+1) dimensions and various remedies have been proposed [21]. We take it as a sign that one should enlarge $\mathcal{A}_c^l$ to some $\bar{\mathcal{A}}_c^l$ and work in the Hilbert space $\bar{\mathcal{H}}$ generated by $\bar{\mathcal{A}}_c^l$ on the natural extension of the state. Thus we add to $\mathcal{A}_c^l$ the idealized element $e^{i2\pi j_{\varphi_\varepsilon}} = U_\pi$ and keep σ and ω_β as before. Equivalently we take the automorphism γ generated by U_π and consider the crossed product $\bar{\mathcal{A}}_c^l = \mathcal{A}_c^l \overset{\gamma}{\rtimes} \mathbf{Z}$. There is a natural extension $\bar{\omega}$ to $\bar{\mathcal{A}}_c^l$ and a natural isomorphism of $\bar{\mathcal{H}}$ and $\bar{\mathcal{A}}_c^l|\bar{\Omega}\rangle$. Here $\bar{\mathcal{H}}$ is the countable orthogonal sum of sectors with n particles created by U_π. Thus,

$$\langle\Omega|e^{ij_f}U_\pi|\Omega\rangle = 0 \tag{3.2}$$

means that U_π leads to the one-particle sector, in general

$$\langle\Omega|U_\pi^{*n}\ e^{ij_f}U_\pi^m|\Omega\rangle = \delta_{nm}\,\omega_\beta(\gamma^n\ e^{ij_f}).$$

The quasifree automorphisms on $\mathcal{A}_c^l$ (e.g. τ_t) can be naturally extended to $\bar{\mathcal{A}}_c^l$, $\tau_t U_\pi = e^{i\pi j_{\varphi_{\varepsilon,t}}}$, $\varphi_{\varepsilon,t}(x) = \varphi_\varepsilon(x+t)$ and since $\varphi_\varepsilon - \varphi_{\varepsilon,t} \in H_1 \ \forall\ t$, this does not lead out of $\bar{\mathcal{A}}_c^l$.

U_π has some features of a fermionic field since

$$\sigma(\varphi_\varepsilon, \tau_t\varphi_\varepsilon) = -\sigma(\varphi_\varepsilon, \tau_{-t}\varphi_\varepsilon) = \frac{1}{4\pi}\left\{ \begin{array}{cl} 1 & \text{for } t > \varepsilon \\ \frac{2t}{\varepsilon} - \frac{t^2}{\varepsilon^2} & \text{for } 0 \leq t \leq \varepsilon \end{array} \right. . \tag{3.3}$$

More generally we could define $U_\alpha = e^{i\sqrt{2\pi\alpha} j_{\varphi_\varepsilon}}$ and get from (3.3) with

$$\operatorname{sgn}(t) = \Theta(t) - \Theta(-t) = \left\{ \begin{array}{rll} 1 & \text{for} & t > 0 \\ 0 & \text{for} & t = 0 \\ -1 & \text{for} & t < 0. \end{array} \right.$$

Proposition (3.4)

$$\begin{aligned} U_\alpha \tau_t U_\alpha &= \tau_t(U_\alpha) U_\alpha\, e^{i\,\alpha\,\operatorname{sgn}(t)/2}, \\ U_\alpha^* \tau_t U_\alpha &= \tau_t(U_\alpha) U_\alpha^*\, e^{i\,\alpha\,\operatorname{sgn}(t)/2} \quad \forall\ |t| > \varepsilon. \end{aligned}$$

Remark (3.5)
We note a striking difference between the general case of anyon statistics and the two particular cases — Bose ($\alpha = 2 \cdot 2n\pi$) or Fermi ($\alpha = 2(2n+1)\pi$) statistics. Only in the latter two cases parity P (2.12:4) is an automorphism of the extended algebra generated through U_α. Thus P which was destroyed in $\mathcal{A}_c^l$ is now recovered for two subalgebras.

The particle sectors are orthogonal in any case

$$\langle \Omega | U^{*n}_\alpha\, e^{ij_f} U^m_\alpha | \Omega \rangle = 0 \quad \forall\ n \neq m,\ f \in H_1.$$

Furthermore, sectors with different statistics are orthogonal $\langle \Omega | U^*_\alpha U_{\alpha'} | \Omega \rangle = 0$, $\alpha \neq \alpha'$, thus if we adjoin U_α, $\forall \alpha \in \mathbf{R}_+$, $\bar{\mathcal{H}}_\beta$ becomes nonseparable.

Next we want to get rid of the ultraviolet cut-off and let ε go to zero. Proceeding the same way we can extend σ and τ_t but keeping ω the sectors abound. The reason is that $\varphi_\varepsilon \stackrel{\varepsilon \to 0}{\longrightarrow} \Theta(x)$ and

$$\|\Theta - \Theta_t\|^2 = \int_{-\infty}^{\infty} \frac{dp\ p}{1 - e^{-\beta p}} \frac{|1 - e^{itp}|^2}{p^2}$$

is finite near $p = 0$ but diverges logarithmically for $p \to \infty$. This means that $e^{ij_f} e^{ij_\Theta} |\Omega\rangle$, $f \in H_1$ gives a sector where one of these particles (fermions, bosons or anyons) is at the point $x = 0$ and is orthogonal to $e^{ij_f} e^{ij_{\Theta_t}} |\Omega\rangle\ \ \forall\ t \neq 0$. Thus the total Hilbert space is not separable and the shift τ_t is not even weakly continuous, so there is no chance to make sense of $\frac{d}{dt}\tau_t e^{ij_\Theta}$.

So far, only one chiral component has been considered. When both chiralities are present, no significant changes arise in the construction described. The only point demanding for some care is the anticommutativity between left- and right- moving fermions,

which asks for an even larger extension of the current algebra by extending its test–functions space.

So, the Weyl algebra $\mathcal{A}_c = \mathcal{A}_c^r \otimes \mathcal{A}_c^l$ is now generated by the unitaries

$$W(f_r, f_l) = e^{i\int (f_r(x)j_r(x)+f_l(x)j_l(x))dx},$$

with $\sigma(f_l, g_l)$ given by (2.11) and $\sigma(f_r, g_r) = -\sigma(f_l, g_l)$. The minimal extension of $\mathcal{A}_c$ is then obtained by adding two idealized elements,

$$U_\pi^l := W(c_l,\, 2\pi(1-\varphi_\varepsilon)) \quad \text{and} \quad U_\pi^r := W(2\pi\varphi_\varepsilon,\, c_r)$$

$$c_l - c_r = (2k+1)\pi, \quad k \in \mathbf{Z} \qquad (\text{e.g. } c_r = \pi/2 = -c_l)$$

They generate for $\varepsilon \to 0$ (not inner) automorphisms of $\mathcal{A}_c$

$$U_\pi^{r(l)}: \quad \gamma_{r(l)}\, W(f_r, f_l) = e^{if_{r(l)}(0)}\, e^{\frac{i}{4}\int_{-\infty}^{\infty} f'_{r(l)}(y)dy}\, W(f_r, f_l)$$

in which for the two subalgebras $\{W(\bar f, \bar f)\}$ and $\{W(\bar f, -\bar f)\}$, $\bar f \in \mathcal{D}_o \subset \mathcal{C}_o^\infty$, the vector and axial gauge transformations can easily be traced back.

In addition to the obvious replacement of (3.4), also the following relation holds

$$U_\pi^{r(*)}\, U_\pi^l = -U_\pi^l\, U_\pi^{r(*)}.$$

Thus we can identify the chiral components of the fermion field with the so constructed unitaries

$$\begin{aligned} \psi_r(x) &= \lim_{\varepsilon\to 0} \exp\{i2\pi \int_{-\infty}^{\infty} \varphi_\varepsilon(y-x) j_r(x')dx' \pm i\frac{\pi}{2}\int_{-\infty}^{\infty} j_l(x')dx'\} \\ \psi_l(x) &= \lim_{\varepsilon\to 0} \exp\{\mp i\frac{\pi}{2}\int_{-\infty}^{\infty} j_r(x')dx' + i2\pi\int_{-\infty}^{\infty} \varphi_\varepsilon(x-y) j_l(x')dx'\} \end{aligned} \tag{3.6}$$

In general, we could define an extension of the algebra $\mathcal{A}_c$ through the abstract elements $U_\alpha^{r(l)}$

$$\begin{aligned} U_\alpha^r &:= W\left(\sqrt{2\pi\alpha}\,\varphi_\varepsilon,\, \frac{1}{2}\sqrt{\frac{\pi\alpha}{2}}\right) \\ U_\alpha^l &:= W\left(-\frac{1}{2}\sqrt{\frac{\pi\alpha}{2}},\, \sqrt{2\pi\alpha}\,(1-\varphi_\varepsilon)\right) \end{aligned}$$

Propositon (3.4) extends also for the non-chiral model generalization. As expected, admitting arbitrary values for α, we get a very rich field structure where definite statistic behaviour is preserved only within a given field class (fixed value of α), so that even different fermions (with different “2 x odd” values of α) do not anticommute but instead follow the general fractional statistics law.

4 Anyon fields in $\pi_{\bar{\omega}}(\bar{\mathcal{A}}_c)''$

Next we shall use another ultraviolet limit to construct local fields which obey some anyon statistics. Of course quantities like

$$[\Psi^*(x),\Psi(x')]_\alpha := \Psi^*(x)\Psi(x')e^{i\frac{2\pi-\alpha}{4}\mathrm{sgn}(x'-x)} + \Psi(x')\Psi^*(x)e^{-i\frac{2\pi-\alpha}{4}\mathrm{sgn}(x'-x)} = \delta_\alpha(x-x')$$

will only be operator valued distributions and have to be smeared to give operators. Furthermore in this limit the unitaries we used so far have to be renormalized so that the distribution $\delta_\alpha(x-x')$ (which should be localized at a point and for certain values of α is supposed to coincide with the ordinary δ-function) gets a factor 1 in front. A candidate for $\Psi(x)$ will be

$$\Psi(x) := \lim_{\varepsilon\to 0} n_\alpha(\varepsilon)\exp\left[i\sqrt{2\pi\alpha}\int_{-\infty}^{\infty} dy\ \varphi_\varepsilon(x-y)j(y)\right]$$

with φ_ε from (3.1) and $n_\alpha(\varepsilon)$ a suitably chosen normalization. With the shorthand $\varphi_{\varepsilon,x}(y) = \varphi_\varepsilon(x-y)$ and $\tilde{\alpha} = \sqrt{2\pi\alpha}$ we can write

$$\begin{aligned}
\Psi^*_\varepsilon(x)\Psi_\varepsilon(x') &= \exp\left\{i\,2\pi\alpha\,\sigma(\varphi_{\varepsilon,x},\varphi_{\varepsilon,x'})\right\}\exp\left\{i\tilde{\alpha}\, j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\},\\
\Psi_\varepsilon(x')\Psi^*_\varepsilon(x) &= \exp\left\{-i\,2\pi\alpha\,\sigma(\varphi_{\varepsilon,x},\varphi_{\varepsilon,x'})\right\}\exp\left\{i\,\tilde{\alpha}\, j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\}.
\end{aligned}$$

We had in (3.3)

$$\begin{aligned}
4\pi\sigma(\varphi_{\varepsilon,x},\varphi_{\varepsilon,x'}) &= \mathrm{sgn}(x-x')\left\{\Theta(|x-x'|-\varepsilon)+\Theta(\varepsilon-|x-x'|)\frac{(x-x')^2}{\varepsilon^2}\right\}\\
&=: \mathrm{sgn}(x-x')D_\varepsilon(x-x')
\end{aligned}$$

and thus

$$[\Psi^*_\varepsilon(x),\Psi_\varepsilon(x')]_\alpha = 2n_\alpha(\varepsilon)^2\cos\left[\mathrm{sgn}(x-x')\left(\frac{\pi}{2}-\frac{\alpha}{4}(1-D_\varepsilon(x-x'))\right)\right]\exp\left\{i\tilde{\alpha}j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\}.$$

Note that for $|x-x'|\geq\varepsilon$ the argument of the cos becomes $\pm\pi/2$, so the α–commutator vanishes, in agreement with (3.4). To manufacture a δ-function for $|x-x'|\leq\varepsilon$ we note that $\cos(...)>0$ and $\omega_\beta(e^{i\alpha j})>0$, so we have to choose $n_\alpha(\varepsilon)$ such that

$$2n^2_\alpha(\varepsilon)\varepsilon\int_{-1}^{1} d\delta\cos\left(\frac{\pi}{2}-\frac{\alpha}{4}(1-\delta^2)\right)\cdot\omega_\beta\left(\exp\left\{i\tilde{\alpha}j_{\varphi_{\varepsilon,x-\varepsilon\delta}-\varphi_{\varepsilon,x}}\right\}\right) = 1$$

and to verify that for $\varepsilon\to 0_+$ $[.,.]_\alpha$ converges strongly to a c-number. For the latter to be finite we have to smear $\Psi(x)$ with L^2-functions g and h:

$$\int dxdx'g^*(x)h(x')[\Psi^*_\varepsilon(x),\Psi_\varepsilon(x')]_\alpha = \int dxdx'g^*(x)h(x')2n_\alpha(\varepsilon)^2\cos(\ )\exp\left\{i\tilde{\alpha}j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\}.$$

This converges strongly to $\langle g|h\rangle$ if for $\varepsilon \to 0_+$

$$\left\langle \exp\left\{-i\tilde{\alpha} j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\} \exp\left\{i\tilde{\alpha} j_{\varphi_{\varepsilon,y'}-\varphi_{\varepsilon,y}}\right\}\right\rangle_\beta - \left\langle \exp\left\{-i\tilde{\alpha} j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\}\right\rangle_\beta \left\langle \exp\left\{i\tilde{\alpha} j_{\varphi_{\varepsilon,y'}-\varphi_{\varepsilon,y}}\right\}\right\rangle_\beta \to 0$$

for almost all x, x', y, y'. Now

$$\langle e^{-ij_a} e^{ij_b}\rangle_\beta = \langle e^{-ij_a}\rangle_\beta \langle e^{ij_b}\rangle_\beta \exp\left[\int_{-\infty}^{\infty} \frac{dp\ p}{1-e^{\beta p}} \widetilde{a}(-p)\widetilde{b}(p)\right].$$

In our case this last factor is

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dp\ p}{1-e^{-\beta p}} \frac{|1-e^{ip\varepsilon}|^2}{\varepsilon^2 p^4} (e^{ipx} - e^{ipx'})(e^{-ipy} - e^{-ipy'}) = \\ = \quad &\int_{-\infty}^{\infty} \frac{dp\ 2(1-\cos p)}{p^3(1-e^{-\beta p/\varepsilon})} (e^{ipx/\varepsilon} - e^{ipx'/\varepsilon})(e^{-ipy/\varepsilon} - e^{-ipy'/\varepsilon}). \end{aligned}$$

For fixed $\beta \neq 0$ and almost all x, x', y, y' this converges to zero for $\varepsilon \to 0$ by Riemann-Lebesgue. Thus a δ-type distribution is recognized, however the particular structure of the singularity we shall discuss later on, already on the basis of the state.

In the same way one sees that $\exp\left\{i\tilde{\alpha} j_{\varphi_{\varepsilon,x}+\varphi_{\varepsilon,x'}}\right\}$ converges strongly to zero and that the $\Psi_{\varepsilon,g}$ are a strong Cauchy sequence for $\varepsilon \to 0$. To summarize we state

Theorem (4.1)
$\Psi_{\varepsilon,g}$ converges strongly for $\varepsilon \to 0$ to an operator Ψ_g which for $\alpha = 2\pi$ satisfies

$$[\Psi_g^*, \Psi_h]_+ = \langle g|h\rangle, \qquad [\Psi_g, \Psi_h]_+ = 0.$$

If *supp* $g <$ *supp* h,

$$\Psi_g^* \Psi_h\, e^{i\frac{2\pi-\alpha}{4}} + \Psi_h \Psi_g^*\, e^{-i\frac{2\pi-\alpha}{4}} = 0 \qquad \forall \alpha.$$

Furthermore we have to verify the claim (1.5) that also for Ψ_g the current j_f induces the local gauge transformation $g(x) \to e^{2i\alpha f(x)} g(x)$. For finite ε we have

$$e^{ij_f} \Psi_{\varepsilon,g} e^{-ij_f} = \Psi_{\varepsilon, e^{i2\pi\alpha\sigma(f,\varphi_\varepsilon)} g}$$

and for $\varepsilon \to 0_+$ we get $\sigma(f, \varphi_\varepsilon) \to \frac{1}{2\pi} f(0)$, so that $\sigma(f, \tau_x \varphi_\varepsilon) = \frac{1}{2\pi} f(x)$.

To conclude we investigate the status of the “Urgleichung” in our construction. It is clear that the product of operator valued distributions on the r.h.s. can assume a meaning only by a definite limiting prescription. Formally it would be

$$\Psi(x)\Psi^*(x)\Psi(x) = [\Psi(x), \Psi^*(x)]_+ \Psi(x) - \Psi^*(x)\Psi(x)^2 = \delta(0)\Psi(x) - 0.$$

From (2.12:5) we know

$$\frac{1}{i}\frac{\partial}{\partial x}\Psi_\varepsilon(x) = \frac{\sqrt{2\pi\alpha}}{2}\, [\bar{\jmath}(x), \Psi_\varepsilon(x)]_+, \qquad \bar{\jmath}(x) = \frac{1}{\varepsilon}\int_{x-\varepsilon}^{x} dy\ j(y).$$

Using

$$j_{\varphi'} e^{ij_\phi} = \frac{1}{i}\frac{\partial}{\partial\alpha} e^{i\frac{\alpha}{2}\sigma(\varphi',\varphi)} e^{ij_{\varphi+\alpha\varphi'}}\Big|_{\alpha=0}$$

one can verify that the limit $\varepsilon \to 0_+$ exists for the expectation value with a total set of vectors and thus gives densely defined (not closable) quadratic forms. They do not lead to operators but we know from (2.7), (2.8) that they define operator valued distributions for test functions from H_1. Thus one could say that in the sense of operator valued distributions the Urgleichung holds

$$\frac{1}{i}\frac{\partial}{\partial x}\Psi(x) = \frac{\sqrt{2\pi\alpha}}{2}\,[j(x), \Psi(x)]_+. \tag{4.2}$$

The remarkable point is that the coupling constant λ in (1.1) is related to the statistics parameter α. For fermions one has a solution only for $\lambda = \sqrt{2\pi}$. Of course one could for any λ enforce fermi statistics by renormalizing the bare fermion field $\psi \to \sqrt{Z}\,\psi$, $j \to Zj$ with a suitable $Z(\lambda)$ but this just means pushing factors around. Alternatively one could extend $\mathcal{A}_c$ by adding $e^{i\sqrt{2\pi\alpha}\,j_{\varphi_\varepsilon}}$, for all $\alpha \in \mathbf{R}_+$. Then one gets in $\mathcal{H}_\omega$ uncountably many orthogonal sectors, one for each α, and in each sector a different Urgleichung holds. Thus different anyons live in orthogonal Hilbert spaces and $e^{i\sqrt{2\pi\alpha}\,j_{\varphi_\varepsilon}}$ is not even weakly continuous in α. If α is tied to λ it is clear that an expansion in λ is doomed to failure and will never reveal the true structure of the theory.

5 Correlation functions in a KMS-state

In terms of distributions, the τ-KMS-state ω_β over $\mathcal{A}^l$ reads

$$\begin{aligned}\langle\psi^*(x)\psi(x')\rangle_\beta &= \int_{-\infty}^{\infty} dp \frac{e^{-ip(x-x')}}{2\pi\,(1+e^{\beta p})} \\ &= -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{i(x-x'+i\varepsilon)-n\beta}\end{aligned} \tag{5.1}$$

It can be also represented in a form that makes the thermal contributions explicit

$$\langle\psi^*(x)\psi(x')\rangle_\beta = \frac{i(x-x')}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(x-x')^2+n^2\beta^2} - \frac{1}{2\pi[i(x-x')-\varepsilon]}$$

or, for a better comparison with the dressed-fermion correlators, as

$$\langle\psi^*(x)\psi(x')\rangle_\beta = \frac{i}{2\beta\,\sinh\frac{\pi(x-x'+i\varepsilon)}{\beta}}\,, \tag{5.2}$$

respectively

$$\langle\psi(x')\psi^*(x)\rangle_\beta = -\frac{i}{2\beta\,\sinh\frac{\pi(x-x'-i\varepsilon)}{\beta}}\,. \tag{5.3}$$

The anyon fields that are present in the v. Neumann algebra $\bar{\pi}_\beta(\bar{\mathcal{A}}_c)''$, associated to the extended current algebra $\bar{\mathcal{A}}_c$, are of the form

$$\Psi_\alpha(x) \sim \exp\left[i\sqrt{2\pi\alpha}\int_{-\infty}^{\infty} dy\; f^x(y) j(y)\right] \tag{5.4}$$

with an appropriately chosen smearing function f (see Sect. 3, 4; $f^x(y) = f(x-y)$).

The fields (5.4) are elements of a Weyl algebra, so they obey the usual multiplication law

$$W(f)W(g) = W(f+g)\, e^{-i\sigma(f,g)/2}\,,$$

with σ — the symplectic form of the algebra. Recalling the symplectic form inherited by $\bar{\pi}_\beta(\bar{\mathcal{A}}_c)''$ from the current algebra $\mathcal{A}_c$ (so, actually from $\pi_\beta(\mathcal{A}_c)''$), (2.11), and with (2.12,3) in mind, the correlation functions of interest can be easily obtained.

We start with the relation

$$\omega(e^{ij_f}e^{ij_g}) = \exp\left\{-\frac{1}{2}\left(\omega(j_f^2) + \omega(j_g^2) + 2\omega(j_f j_g)\right)\right\} \tag{5.5}$$

with

$$\omega_\beta(j_f j_g) = -\int \frac{dydy' f(y)g(y')}{(2\beta)^2 \sinh^2 \frac{\pi(y-y'+i\varepsilon)}{\beta}}\,, \qquad \varepsilon \to 0_+\,.$$

The shift $\tau_t\,:\; f(y) \to f(y-t)$ can be continued analytically for f in the upper strip $0 < \mathrm{Im}\; t < \beta$ and for g in the lower strip $0 > \mathrm{Im}\; t > -\beta$. This reflects the KMS-property of ω_β and also serves as an ultraviolet cut-off if we consider ε as arising from $\tau_{i\varepsilon} j_f$. We are interested in $f^x(y) \sim -\Theta(x-y)$, $g^{x'}(y') \sim \Theta(x'-y')$ and then an infrared cut-off $\Theta(-y) \to \Theta(-y-\delta)$ will cancel out but to get rid of the ultraviolet cut-off we have to define

$$\Psi(x) = \lim_{\varepsilon\to 0} n_\alpha(\varepsilon) \exp\left\{i\sqrt{2\pi\alpha}\, j_{\Theta_\varepsilon^x}\right\} =: \Psi_\alpha(x)\,. \tag{5.6}$$

In the expectation values in (5.5) the limits of the integrals are $\int_{-\delta}^{x}\int_{-\delta}^{x'}$, $\int_{-\delta}^{x}\int_{-\delta}^{x}$ and $\int_{-\delta}^{x'}\int_{-\delta}^{x'}$ for $j_f j_g$, j_f^2 and j_g^2 respectively. We only have to work out the first, the others are special cases, and we can scale π/β away:

$$\int_{-\delta}^{x} dy \int_{-\delta}^{x'} dy' \sinh^{-2}(y-y'+i\varepsilon) = \ln\frac{\sinh(x-x'+i\varepsilon)}{\sinh(x+\delta+i\varepsilon)} - \ln\frac{\sinh(-\delta-x'+i\varepsilon)}{\sinh(i\varepsilon)}\,.$$

To evaluate (5.5), we have to subtract from this expression 1/2 the same with $x' = x$ and 1/2 the same with $x = x'$. Then for $\delta \to \infty$ the δ-terms cancel out but what remains (already with the proper coefficient) is

$$(\alpha/2\pi)\left[-\ln\sinh\frac{\pi(x-x'+i\varepsilon)}{\beta} + \ln(i\varepsilon)\right]\,.$$

Upon exponentiation we get for $\alpha = 2\pi$ the fermion KMS two-point function and a factor ε which has to be compensated by the renormalization of Ψ, that is for fermions

the renormalization parameter should be $n_{2\pi}(\varepsilon) = (2\pi\varepsilon)^{-1/2}$ and for the general case of an arbitrary α: $n_\alpha(\varepsilon) = (2\pi\varepsilon)^{-\alpha/4\pi}$.

For all α's the two-point function (for $x \neq x'$ and $\beta = \pi$)

$$\langle \Psi^*_\alpha(x)\Psi_\alpha(x')\rangle_\beta = \langle \Psi_\alpha(x)\Psi^*_\alpha(x')\rangle_\beta = \left(\frac{i}{2\beta\sinh(x-x')}\right)^{\alpha/2\pi} =: S_\alpha(x-x') \tag{5.7}$$

has the desired properties

1. Hermiticity:

$$S^*_\alpha(x) = S_\alpha(-x) \iff \langle \Psi^*_\alpha(x)\Psi_\alpha(x')\rangle^*_\beta = \langle \Psi^*_\alpha(x')\Psi_\alpha(x)\rangle_\beta\,;$$

2. α-commutativity:

$$S_\alpha(-x) = e^{i\alpha/2}S_\alpha(x) \iff \langle \Psi_\alpha(x')\Psi^*_\alpha(x)\rangle_\beta = e^{i\alpha/2}\langle \Psi^*_\alpha(x)\Psi_\alpha(x')\rangle_\beta\,;$$

3. KMS-property:

$$S_\alpha(x) = S_\alpha(-x+i\pi) \iff \langle \Psi^*_\alpha(x)\Psi_\alpha(x')\rangle_\beta = \langle \Psi_\alpha(x')\Psi^*_\alpha(x+i\pi)\rangle_\beta\,.$$

Remark
If the deformation parameter $\exp i\alpha/2 = q \notin \mathbf{S}^1$ but instead $q \in (-1,1)$ then there are no KMS-states since there do not exist even translation invariant states [22].

For $\alpha = 2\pi$ Eq.(5.7) reads

$$\langle \Psi^*_{2\pi}(x)\Psi_{2\pi}(x')\rangle_\beta = \lim_{\varepsilon\to 0_+} \frac{i}{2\beta\,\sinh\frac{\pi(x-x'+i\varepsilon)}{\beta}}\,. \tag{5.8}$$

For $f(y) \to \Theta(x-y)$, $g(y') \to -\Theta(x'-y')$ nothing changes so we verify the realtion (5.2) $\leftrightarrow$ (5.3),

$$\langle \Psi^*_{2\pi}(x)\Psi_{2\pi}(x')\rangle_\beta = \langle \Psi_{2\pi}(x)\Psi^*_{2\pi}(x')\rangle_\beta\,.$$

For $\alpha = 4\pi$ we get like for the j's

$$\langle \Psi^*_{4\pi}(x)\Psi_{4\pi}(x')\rangle_\beta = -\frac{1}{\left(2\beta\,\sinh\frac{\pi(x-x'+i\varepsilon)}{\beta}\right)^2}\,, \tag{5.9}$$

whereas for $\alpha = 6\pi$ we get a different kind of fermions

$$\langle \Psi^*_{6\pi}(x)\Psi_{6\pi}(x')\rangle_\beta = -\frac{i}{\left(2\beta\,\sinh\frac{\pi(x-x'+i\varepsilon)}{\beta}\right)^3}\,. \tag{5.10}$$

They are not canonical fields since

$$-[\Psi^*_{6\pi}(x),\Psi_{6\pi}(x')]_\alpha = [\Psi^*_{6\pi}(x),\Psi_{6\pi}(x')]_+ = -\frac{1}{8\pi^2}\left(\delta''(x-x') - \frac{\pi^2}{\beta^2}\delta(x-x')\right)\,.$$

This shows that local anticommutativity alone does not guarantee the uniqueness of the KMS-state, one needs in addition the CAR-relations. The Ψ_α's, $\alpha \in (2\mathbf{N}+1)\pi$, describe an infinity of inequivalent fermions.

For the thermal expectation of the α-commutator one finds

$$\begin{aligned}\omega_\beta([\Psi^*(x), \Psi(x')]_\alpha) &= -i\left(-\frac{1}{2\beta\,\sinh\frac{\pi(x-x'+i\varepsilon)}{\beta}}\right)^{\alpha/2\pi} \\ &+ \; i\left(-\frac{1}{2\beta\,\sinh\frac{\pi(x-x'-i\varepsilon)}{\beta}}\right)^{\alpha/2\pi} \end{aligned} \tag{5.11}$$

and for $\varepsilon \to 0$ a distribution is obtained where only the leading singularity is temperature independant, namely

$$\lim_{\varepsilon\to 0}\left\{\left(\frac{1}{x-x'-i\varepsilon}\right)^{\alpha/2\pi} - \left(\frac{1}{x-x'+i\varepsilon}\right)^{\alpha/2\pi}\right\} = ?$$

We shall not attempt to make sense of the “canonical” anyon commutators. Another interesting topic — the higher correlation functions, since the limiting procedures involved might influence their properties, we shall also discuss separately [23].

6 Internal symmetries

We shall briefly discuss what happens in the the case of a fermion multiplet. Then,

$$\{\psi_i^*(x), \psi_k(y)\} = \delta_{ik}\,\delta(x-y), \qquad i = 1, 2, ..., N$$

The CAR-algebra so defined posesses an obvious $U(N)$ symmetry

$$\psi \longrightarrow U\,\psi, \quad U \in U(N)$$

In analogy with the previous case we construct quadratic forms $j_k(x) = \psi_k^*(x)\psi_k(x)$ which satisfy anomalous commutation relations:

$$[j_k(x), j_m(y)] = -\frac{i}{2\pi}\delta_{km}\delta'(x-y) \tag{6.1}$$

and give rise to operators $j_{f_k} = \int j_k(x) f_k(x) dx,\ f \in H_1$.

Denote the corresponding Weyl operators with $W(f_1, ..., f_N)$

$$W(f_1, ..., f_N) := \exp\{i\sum_{n=1}^{N} j_{f_n}\}$$

The current algebra (6.1) has a (global) $O(N)$ symmetry,

$$j \to Mj = j', \qquad M \in O(N) \tag{6.2}$$

Genuine anticommuting fields can now be identified in an extension of the current algebra quite similar to the one for the non-chiral one-colour case. More precisely, we have to allow for the existence in the new algebra of the following elements

$$U_{\pi_k} = e^{i2\pi \int\limits_{-\infty}^{\infty} \varphi(x-x')j_k(x')dx' + i \sum\limits_{n=1\,,n\neq k}^{N} c_n \int\limits_{-\infty}^{\infty} j_n(x')dx'} =: \psi_k(x),$$

$$c_k \in \mathbf{R}, \quad c_k - c_n = (2l+1)\pi, \quad l \in \mathbf{Z},\ \forall k,n\,. \tag{6.3}$$

For the elements so defined the following relation holds

$$\psi_k^*(x)\psi_l(y) = \psi_l(y)\psi_k^*(x)e^{-i\pi\,\delta_{kl}\,\mathrm{sgn}(y-x)}\,e^{-i(1-\delta_{kl})(c_k-c_l)}$$

which would then lead (after an appropriate renormalisation) to the desired CAR's.

How should one consider the element, obtained through the same ansatz, but after a transformation (6.2) of the currents, i.e.

$$\psi_k'(x) = e^{i2\pi\int_{-\infty}^{x} M_{kl}j_l(x')dx' + (\ldots)} \tag{6.4}$$

Has it something to do with the $U(N)$-transformed fermion ψ_k', i.e.

$$e^{i2\pi\int_{-\infty}^{x} M_{kl}j_l(x')dx'} \stackrel{?}{\longleftrightarrow} U_{kl}\psi_l(x)$$

What one notices is that the $O(N)$-transformation, being (of course) an isomorphism of the algebra $\bar{\mathcal{A}}_c$, is no longer an automorphism of the total algebra.

In such a symmetry nonpreservation a major limitation of the usual bosonization scheme (with non-local fermi fields, [24, 25]) has been recognized, e.g. the $O(N)$ symmetry of a free scalar theory with N fields, which emerges upon bosonization of a theory with N free Dirac fields with $U(N)\times U(N)$ chiral invariance, does not correspond to any subgroup of the fermion symmetry group [26]. The bosonization scheme proposed therein preserves all symmetries through the passage from one formulation to the other. However, this does not concern the complete cycle where, as we have seen, symmetries may be (actually, necessarily are) spontaneously destroyed.

We are faced with the similar situation also for the U-invariance we mentioned at the beginning. Here, e.g. for $U(1)$, so

$$\psi_k(x) \to e^{i\alpha}\psi_k(x)$$

we get

$$\psi_k'(x) = e^{i\alpha}e^{i2\pi\int_{-\infty}^{x} j_k(x')dx'} = e^{i2\pi\int_{-\infty}^{x} j_k'(x')dx'}$$

with

$$j_k'(x') = j_k(x') + \frac{1}{2\pi}\bar{\alpha}(x'), \quad \bar{\alpha}_{(k)} : \int_{-\infty}^{x} \bar{\alpha}(x') = \alpha(x)$$

In the local case this still remains an automorphism of the extended algebra, which is in agreement with the very idea of the construction presented, while for a global

transformation this is no longer the case, so for the total algebra so constructed there is no global $U(1)$ symmetry present.

However, on the passage from the CAR-algebra $\mathcal{A}$ to the current algebra $\mathcal{A}_c$ contained in the v.Neumann algebra $\pi_\beta(\mathcal{A})''$ the parity has been broken, but also as an isomorphism of $\mathcal{A}_c$.

Thus we are in a situation in which a particular (physically motivated) extension $\bar{\mathcal{A}}_c$, of the algebra of observables (the current algebra $\mathcal{A}_c$) is constructed, $\bar{\mathcal{A}}_c \supset \mathcal{A}_c$, such that $\nexists \beta \in \mathrm{Aut}\, \bar{\mathcal{A}}_c : \; \beta|_{\mathcal{A}} = \alpha$ for some $\alpha \in \mathrm{Aut}\, \mathcal{A}_c$. This phenomenon we call *symmetry destruction*. It is related to the spontaneous collapse of a symmetry, discussed by Buchholz and Ojima in the context of supersymmetry [16] and is seen to be a field effect since in contrast to the spontaneous symmetry breaking [17], it cannot occur in a finite-dimensional Hilbert space.

7 Concluding remarks

To summarize we gave a precise meaning to eq.(1.2a,b,c) by starting with bare fermions, $\mathcal{A} = \mathrm{CAR}(\mathbf{R})$. The shift τ_t is an automorphism of $\mathcal{A}$ which has KMS-states ω_β and associated representations π_β. In $\pi_\beta(\mathcal{A})''$ one finds bosonic modes $\mathcal{A}_c$ with an algebraic structure independent on β. Taking the crossed product with an outer automorphism of $\mathcal{A}_c$ or equivalently augmenting $\mathcal{A}_c$ by an unitary operator to $\bar{\mathcal{A}}_c$ we discover in $\bar{\pi}_\beta(\bar{\mathcal{A}}_c)''$ anyonic modes which satisfy the Urgleichung in a distributional sense. For special values of λ they are dressed fermions distinct from the bare ones. From the algebraic inclusions $\mathrm{CAR}(\mathit{bare}) \subset \pi_\beta(\mathcal{A})'' \supset \mathcal{A}_c \subset \bar{\mathcal{A}}_c \subset \bar{\pi}_\beta(\bar{\mathcal{A}}_c)'' \supset \mathrm{CAR}(\mathit{dressed})$ one concludes that in our model it cannot be decided whether fermions or bosons are more fundamental. One can construct the dressed fermions either from bare fermions or directly from the current algebra. Also the correlation functions in the temperature state in both cases coincide that is in agreement with the uniqueness of the τ-KMS state over a CAR algebra (τ being the shift automorphism). The corresponding anyonic two-point function evaluated for the special case of a scalar field reproduces the recent result due to Borchers and Yngvason.

Acknowledgements

The authors are grateful to H. Narnhofer, R. Haag and J. Yngvason for helpful discussions on the subject of this paper.

N.I. acknowledges the financial support from the "Fonds zur Förderung der wissenschaftlichen Forschung in Österreich" under grant P11287–PHY and the hospitality at the Institute for Theoretical Physics of University of Vienna.

Appendix The field algebra as a crossed product

The idea that the crossed product C^*-algebra extension is the tool that makes possible construction of fermions (so, unobservable fields) from the observable algebra has been first stated in [27]. There, the problem of obtaining different field groups has been shown to amount to construction of extensions of the observable algebra by the group duals. Explicitly, crossed products of C^*-algebras by semigroups of endomorphisms have been introduced when proving the existence of a compact global gauge group in particle physics given only the local observables [28]. Also in the structural analysis of the symmetries in the algebraic QFT [5] extendibility of automorphisms from a unital C^*-algebra to its crossed product by a compact group dual becomes of importance since it provides an analysis of the symmetry breaking [17] and in the case of a broken symmetry allows for concrete conlusions for the vacuum degeneracy [29].

The reason why a relatively complicated object — crossed product over a specially directed symmetric monoidal subcategory End $\mathcal{A}$ of unital endomorphisms of the observable algebra $\mathcal{A}$, is involved in considerations in [29] is that in general, non-Abelian gauge groups are envisaged. For the Abelian group $U(1)$ a significant simplification is possible since its dual is also a group — the group $\mathbf{Z}$. On the other hand, even in this simple case the problem of describing the local gauge transformations remains open. Therefore in the Abelian case consideration of crossed products over a discrete group offers both a realistic framework and reasonable simplification for the analysis of the resulting field algebra. We shall briefly outline the general construction for this case, for more details see [13].

We start with the CCR algebra $\mathcal{A}(\mathcal{V}_0,\sigma)$ over the real symplectic space $\mathcal{V}_0$ with symplectic form σ, Eq.(2.11), generated by the unitaries $W(f)$, $f\in\mathcal{V}_0$ with

$$W(f_1)W(f_2)=e^{-i\sigma(f_1,f_2)/2}W(f_1+f_2),\qquad W(f)^*=W(-f)=W(f)^{-1}.$$

Instead of the canonical extension $\bar{\mathcal{A}}(\mathcal{V},\bar\sigma)$, $\mathcal{V}_0\subset\mathcal{V}$ [12], we want to construct another algebra $\mathcal{F}$, such that $\mathrm{CCR}(\mathcal{V}_0)\subset\mathcal{F}\subset\mathrm{CCR}(\mathcal{V})$ and we choose $\mathcal{V}_0=\mathcal{C}_0^\infty$, $\mathcal{V}=\partial^{-1}\mathcal{C}_0^\infty$. Any free (not inner) automorphism α, $\alpha\in\mathrm{Aut}\,\mathcal{A}$ defines a crossed product $\mathcal{F}=\mathcal{A}\overset{\alpha}{\rtimes}\mathbf{Z}$. This may be thought as (see [30]) adding to the initial algebra $\mathcal{A}$ a single unitary operator U together with all its powers, so that one can formally write $\mathcal{F}=\sum_n\mathcal{A}\,U^n$, with U implementing the automorphism α in $\mathcal{A}$, $\alpha A=U\,A\,U^*$, $\forall A\in\mathcal{A}$. Operator U should be thought of as a charge-creating operator and $\mathcal{F}$ is the minimal extension — an important point in comparison to the canonical extension which we find superfluous especially when questions about statistical behaviour and time evolution are to be discussed. With the choice

$$\alpha W(f)=e^{i\sigma(\bar g,f)}W(f),\qquad \bar g\in\mathcal{V}\backslash\mathcal{V}_0,\qquad \mathcal{V}_0\subset\mathcal{V}\tag{A.1}$$

and identifying $U=W(\bar g)$, $\mathcal{F}$ is in a natural way a subalgebra of $\mathrm{CCR}(\mathcal{V})$.

If we take for $\mathcal{A}$ the current algebra $\mathcal{A}_c$ and for U — the idealized element U_π to be added to it, we find an obvious correspondence between the functional picture from Sec.3 and the crossed product construction. However, in the latter there is an additional

structure present which makes it in some cases favourable. Writing an element $F \in \mathcal{F}$ as $F = \sum_n A_n U^n, \quad A_n \in \mathcal{A}$, we see that it is convenient to consider $\mathcal{F}$ as an infinite vector space with U^n as its basic unit vectors and $A_n =: (F)_n$ as components of F. The algebraic structure of $\mathcal{F}$ implies that multiplication in this space is not componentwise but instead

$$(F.G)_m = \sum_n F_n \, \alpha^n \, G_{m-n}.$$

Given a quasifree automorphism $\rho \in \mathrm{Aut}\,\mathcal{A}$, it can be extended to $\mathcal{F}$ if and only if the related automorphism $\gamma_\rho = \rho\alpha\rho^{-1}\alpha^{-1}$ is inner for $\mathcal{A}$. Since γ_ρ is implemented by $W(\bar{g}_\rho - \bar{g})$, this is nothing else but demanding that $\bar{g}_\rho - \bar{g} \in \mathcal{V}_0$ and this is exactly the same requirement as in the functional picture. This appears to be the case for the space translations and also for the time evolution, but in the absence of long-range forces [13].

Also a state $\omega(.)$ over $\mathcal{A}$ together with the representation π_ω associated with it through the GNS-construction can be extended to $\mathcal{F}$. The representation space of $\mathcal{F}$ can be regarded as a direct sum of charge-n subspaces, each of them being associated with a state $\omega \circ \alpha^{-n}$ and with $\mathcal{H}_0$, the representation space of $\mathcal{A}$, naturally imbedded in it. Since ω is irreducible and $\omega \circ \alpha^{-n}$ not normal with respect to it, the extension of the state over $\mathcal{A}$ to a state over $\mathcal{F}$ is uniquely determined by the expectation value with $|\Omega_0\rangle = |\omega\rangle$ in this representation

$$\langle \Omega_k | W^*(f) W(h) W(f) | \Omega_n \rangle = \delta_{kn} \, e^{-i\sigma(f + n\bar{g}, h)} \omega(W(h))$$

where $U_k|\Omega\rangle := |\Omega_k\rangle$, $\langle \Omega_k | \Omega_n \rangle = \delta_{kn}$. This states nothing but orthogonality of the different charge sectors, the same as in the functional description, Eq.(3.2).

In the crossed product gauge automorphism is naturally defined with

$$\gamma_\nu \, U^n = e^{2\pi i \nu n} U^n, \qquad \gamma_\nu \, W(f) = W(f)\,. \tag{A.2}$$

Thus for the representation π_Ω one finds

$$\gamma_\nu \left(|F(f)^{(k)}\rangle \right) = \gamma_\nu \left(W(f) |\Omega_k\rangle \right) = e^{2\pi i \nu k} W(f) |\Omega_k\rangle,$$

that justifies interpretation of the vectors $|F(f)^{(k)}\rangle$ as belonging to the charge-k subspace. However, $\mathcal{A}$ is a subalgebra of $\mathcal{F}$ for the gauge group $\mathcal{T} = [0, 1)$, while it is a subalgebra of CAR for the gauge group $\mathcal{T} \otimes \mathbf{R}$. Thus the crossed product algebra so constructed, being really a Fermi algebra, does not coincide with CAR but is only contained in it. In other words, such a type of extension does not allow incorporation also of local gauge transformations which are of main importance in QFT.

Therefore we need a generalization of the construction in [13] which would describe also the local gauge transformations. The most natural candidate for a structural automorphism would be

$$\alpha_{\bar{g}_x} W(f) = e^{i \sum_{n=0}^{K} f^{(n)}(x)} W(f). \tag{A.3}$$

However, it turns out that only for $n = 0$ the crossed product algebra so obtained allows for extension of space translations as an automorphism of $\mathcal{A}$ — the minimal requirement

one should be able to meet. Already first derivative gives for the zero Fourier component of the difference $\bar{g}_{x_\delta} - \bar{g}_x$ an expression of the type $\int y^{-1}\delta(y)dy$, so it drops out of $\mathcal{C}_0^\infty$. So, the automorphism of interest reads

$$\alpha_{\bar{g}_x} W(f) = e^{if(x)} W(f) \tag{A.4}$$

and can be interpreted as being implemented by $W(\bar{g}_x)$ with $\bar{g}_x = 2\pi\,\Theta(x-y)$. Correspondingly, the operator we add to $\mathcal{A}$ through the crossed product is

$$U_x = e^{i2\pi \int_{-\infty}^{x} j(y)dy}. \tag{A.5}$$

Compared to [13] this means an enlargment of the test functions space not with a kink but with its limit — the sharp step function. In a distributional sense it still can be considered as an element of $\partial^{-1}\mathcal{V}_0$ for some $\mathcal{V}_0$ since the derivative of $\bar{g}_x$ has bounded zero Fourier component. Similarly, the extendibility condition for space translations is found to be satisfied, $\bar{g}_{x_\delta} - \bar{g}_x \in \mathcal{V}_0$ so that in the crossed product shifts are given by

$$\bar{\tau}_{x_\delta} U_x = V_{x_\delta} U_x, \qquad V_{x_\delta} = W(\bar{g}_{x_\delta} - \bar{g}_x). \tag{A.6}$$

Note that shifts do not commute with the structural automorphism $\alpha_{\bar{g}_x}$, $\tau_{x_\delta}\alpha_{\bar{g}_x} W(f) \neq \alpha_{\bar{g}_x}\tau_{x_\delta} W(f)$. Since

$$\sigma(\bar{g}_x, \bar{g}_{x_\delta}) = -\pi \mathrm{sgn}(\delta), \tag{A.7}$$

already the elements of the first class are anticommuting and we identify $U_x =: \psi(x)$. Then (A.4) (after smearing with a function from $\mathcal{C}_0^\infty$) is nothing else but (1.5), i.e. the statement (or requirement) that currents generate local gauge transformations of the so–constructed field. Any scaling of the function which defines the structural automorphism $\alpha_{\bar{g}_x}$ destroys relation (A.7) and fields obeying fractional statistics are obtained instead. This is effectively the same as adding to the algebra $\mathcal{A}$ the element U_α with $\alpha = 2\pi\mu$, μ being the scaling parameter.

However, the crossed product offers one more interesting possibility: when for the symplectic form in question instead of (A.7) (or its direct generalization $\sigma(\bar{g}_x, \bar{g}_{x_\delta}) = (2n+1)\pi, n \in \mathbf{Z}$) another relation takes place, $\sigma(\bar{g}_x, \bar{g}_{x_\delta}) = (2n+1)/\bar{n}^2$ for some fixed $\bar{n} \in \mathbf{Z}$, the crossed product acquires a zone structure, with $2n\bar{n}$–classes commuting, $(2n+1)$–classes anticommuting and elements in the classes with numbers $m \in \mathbf{Z}/\mathbf{Z}_{\bar{n}}$ obeying an anyon statistics with parameter $r = \sqrt{2n+1}\, m/\bar{n}$. So, fields with different statistical behaviour are present in the same algebra, however the Hilbert space remains separable.

We want to emphasize that relation of the type $\psi(x+\delta_x) = U_{x+\delta_x}$ may be misleading, the latter element exists in the crossed product only by Eq.(A.6), so that for the derivative one finds

$$\begin{aligned}\frac{\partial\psi(x)}{\partial x} &:= \lim_{\delta_x\to 0} \frac{\psi(x+\delta_x) - \psi(x)}{\delta_x} = \lim_{\delta_x\to 0} \frac{1}{\delta_x}\left(V_{x_\delta}U_x - U_x\right) = \\ &\lim_{\delta_x\to 0} \frac{1}{\delta_x}\left(e^{i\,2\pi\,\delta_x\, j(x)} - 1\right) U_x = 2\pi\, i\, j(x) U_x =: 2\pi\, i\, j(x)\psi(x).\end{aligned} \tag{A.8}$$

This, together with (2.5) gives for the operators

$$i\psi_{f'} = \psi_f \, j_{\Theta'}. \tag{A.9}$$

Note that in the crossed product, which can actually be considered as a left $\mathcal{A}$-module, equations of motion (A.8), (A.9) appear (due to this reason) without an antisymmetrization, which was the case with the functional realization, Eq.(4.2), but otherwise the result is the same. Therefore the scaling sensitivity of the crossed product field algebra is another manifestation of the quantum “selection rule” for the value of λ in the Urgleichung (1.2c).

References

[1] T.D. Lee, *Phys. Rev.* **95**, 1329 (1954).

[2] H. Bethe, *Z. Physik* **71**, 205 (1931).

[3] W. Heisenberg, *Z. Naturforsch.* **9a**, 292 (1954).

[4] W. Thirring, *Ann. Phys.* **3**, 91 (1958).

[5] R. Haag, *Local Quantum Physics* (Springer, Berlin Heidelberg, 1993).

[6] P. Jordan, *Z. Phys.* **93**, 464 (1935); **98**, 759 (1936);**99**, 109 (1936).

[7] M. Born, N. Nagendra–Nath, *Proc. Ind. Acad. Sci.* **3**, 318 (1936).

[8] G. Dell'Antonio, Y. Frischman, D. Zwanziger, *Phys. Rev.* **D6**, 988 (1972).

[9] S. Mandelstam, *Phys. Rev.* **D11**, 3026 (1975).

[10] A.L. Carey, S.N.M. Ruijsenaars, *Acta Appl. Math.* **10**, 1 (1987).

[11] A.L. Carey, C.A. Hurst, D.M. O'Brien, *J. Math. Phys.* **24**, 2212 (1983).

[12] F. Acerbi, G. Morchio, F. Strocchi, *Lett. Math. Phys.* **26** (1992) 13; **27**, 1 (1993).

[13] N. Ilieva, H. Narnhofer, *ÖAW Sitzungsber. II* **205**, 13 (1996).

[14] N. Ilieva, W. Thirring, *Eur. Phys. J.* **C6**, 705 (1999).

[15] H.J. Borchers, J. Yngvason, *J. Math. Phys.* **40**, 601 (1999).

[16] D. Buchholz and I. Ojima, *Nucl. Phys.* **B498**, 228 (1997).

[17] H. Narnhofer, W. Thirring, *Ann. Inst. Henri Poincaré* **70**, 1 (1999).

[18] J. Schwinger, *Phys. Rev.* **128**, 2425 (1962).

[19] H. Grosse, W. Maderner, C. Reitberger, in *Generalized Symmetries in Physics*, Eds. H.–D. Döbner, V. Dobrev, and A. Ushveridze (World Scientific, Singapore, 1994).

[20] D.C. Mattis and E. Lieb, *J. Math. Phys.* **6**, 304 (1965).

[21] F. Strocchi, *General Properties of QFT, Lecture Notes in Physics – vol.51* (World Scientific, Singapore, 1993).

[22] H. Narnhofer, *private communication.*

[23] N. Ilieva, W. Thirring, *in preparation.*

[24] S. Coleman, *Phys. Rev.* **D11**, 2088 (1975).

[25] S. Mandelstam, *Phys. Rev.***D11**, 3026 (1975).

[26] E. Witten, *Commun. Math. Phys.* **92**, 455 (1984).

[27] S. Doplicher, R. Haag, J. Roberts, *Commun. Math. Phys.* **15**, 173 (1969).

[28] S. Doplicher, J. Roberts, *Ann. Math.* **130**, 75 (1989).

[29] D. Buchholz, S. Doplicher, R. Longo, J. Roberts, *Commun. Math. Phys.* **155**, 123 (1993).

[30] H. Araki, in *On Klauder's Path: A Field Trip*, Eds. G. Emch, G. Hegerfeldt, L. Streit (World Scientific, Singapore, 1994), p.11.